Plasma metric singularities in helical devices and tearing instabilities in tokamaks

by

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Abstract

Plasma toroidal metric singularities in helical devices and tokamaks, giving rise to magnetic surfaces inside the plasma devices are investigated in two cases. In the first we consider the case of a rotational plasma on an helical device with circular cross-section and dissipation. In this case singularities are shown to place a Ricci scalar curvature bound on the radius of the surface where the Ricci scalar is the contraction of the constant Riemannian curvature tensor of magnetic surfaces. An upper bound on the initial magnetic field in terms of the Ricci scalar is obtained. This last bound may be useful in the engineering construction of plasma devices in laboratories. The normal poloidal drift velocity is also computed. In the second case a toroidal metric is used to show that there is a relation between singularities and the type of tearing instabilities considered in the tokamak. Besides, in this case Ricci collineations and Killing symmetries are computed.The pressure is computed by applying these constraints to the pressure equations in tokamaks. **PACS numbers:**

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I Introduction

Geometrical techniques have been used with great success [1] in Einstein general relavity have been also used in other important areas of physics, such as plasma structures in tokamaks as been clear in the seminal book by Mikhailovskii [2] to investigate the tearing and other sort of instabilities in confined plasmas [3], where the Riemann metric tensor plays a dynamical role interacting with the magnetic field through the magnetohydrodynamical equations (MHD). Recentely Garcia de Andrade [4] has applied Riemann metric to investigate magnetic flux tubes in superconducting plasmas. Thiffault and Boozer [5] have also applied the methods of Riemann geometry in the context of chaotic flows and fast dynamos. In this paper we use the tools of Riemannian geometry, also used in other branches of physics as general relativity [6], such as Killing symmetries , Riemann and Ricci [7] collineations , shall be applied here to generate magnetic nested surfaces in helical devices and tokamaks through the built of Einstein spaces obtained from Tokamak plasma metric. This work is motivated by the fact that the magnetic surfaces are severely constrained in tokamaks [3] and Killing symmetries are well applied every time we have symmetries in the problem as in solutions of Einstein equations of general relativity. Equilibrium of these surfaces or their instabilities are fundamental in the constructio of tokamaks and other plasma devices such as stellarators where torsion is also present. The magnetic surfaces are more easily obtained when symmetries are present. This is our main motivation to apply the special Riemann geometrical techniques of Killing symmetries and Ricci collineation to obtain magnetic surfaces formed by Einstein spaces. To simplify matters we shall consider two usual approximations from plasma physics [3] which are the small toroidality or inverse aspect ratio $\epsilon = \frac{a}{R} << 1$ where here R represents the external radius of the torus and a is its internal radius, and the Shafranov displacement $\Delta' << 1$ as well. We consider examples of two plasma metrics: The first is the metrical of plasma rotation in tokamaks where plasma dissipation is taken into account. This metric was used for the first time by Tsypin et al [8] to describe dissipative plasmas in the circular cross-section helical devices such as HELIACS or Drakon (stellarator) [9]. The pressure of the tokamak is obtained from the tokamak Shafranov shift equation. Constant pressure closed ergodic nested surfaces in magnetohydrostatics have also been shown by Schief [10] to be generated by solitons. This is another mathematical technique , distinct from ours, is another use ofmathematical theory to generate of nested magnetic surfaces in plasmas. The paper is organised as follows: In section 2 we review the Ricci tensor techniques and Ricci colineations which are not usually familiar to the plasma physicists. In section 3 we solve the Ricci tensor components from the plasma metric by considering that nested surfaces are formed by Einstein spaces, where the Ricci tensor is proportional to the metric. In section 4 stablishing a geometrical method for the classification of tearing instabilities and solve the Ricci collineation equations to find out the Killing vectors for the dissipative rotational plasma metric.. Conclusions are presented in section 5.

II Ricci collineations from plasma metrics

Ricci tensor R_{ik} is contructed from the contraction of Riemann tensor R_{ijkl} in terms of the contravariant components of the metric by the expresion

$$R_{ik} = g^{jl} R_{ijkl} \tag{II.1}$$

and the Ricci scalar R by an extra contraction in terms of the metric tensor as $R = g^{ij} R_{ij}$. Let us now compute the Riemann space of constant curvature represented by the Riemann tensor components

$$R_{ijkl} = \Lambda(g_{ik}g_{jl} - g_{il}g_{jk}) \tag{II.2}$$

where Λ is a constant which is called de Sitter cosmological constant. Contraction of expression (II.2) in two non-consecutive indices,otherwise the symmetry of the Riemann curvature tensor $R_{ijkl} = -R_{jikl} = R_{jilk}$ would make them vanish, yields the Einstein space Ricci relation

$$R_{ik} = 2\Lambda g_{ik} \tag{II.3}$$

The Ricci collineations equations are given by

$$[\partial_l R_{ik}]\eta^l + R_{il}\partial_k \eta^l + R_{kl}\partial_i \eta^l = 0 \tag{II.4}$$

where η^l are the components of the Killing vector $\vec{\eta}$ which defines the symmetries of the associated space, and $\partial_l := \frac{\partial}{\partial x^l}$ are the components of the partial derivative operator. This equation is obtained from the more elegant definition in terms of the Lie derivative $\mathcal{L}_\eta$ as

$$\mathcal{L}_\eta R_{ik} = 0 \tag{II.5}$$

In the next section we shall construct the magnetic surface as plasma metric singularities and in section IV we solve the Ricci collineation equations in terms of the plasma metric above.

III Rotational Plasmas Metric Singularities in Helical Devices

Let us now consider the application of the Tsypin et al metric of a rotational toroidal dissipative plasma tokamak, given by the nonvanishing components

$$g_{11} = \frac{1}{4\pi B_0 \phi} \tag{III.6}$$

$$g_{22} = \frac{\phi}{\pi B_0} \tag{III.7}$$

$$g_{23} = R\tau_0 \frac{\phi}{\pi B_0} \tag{III.8}$$

$$g_{13} = \frac{1}{2\pi {B_0}^{\frac{1}{2}}} \partial_\eta \left(\frac{1}{{B_0}^{\frac{1}{2}}}\right) \tag{III.9}$$

$$g_{33} = R^2 [1 - (\frac{\phi}{\pi B_0})^{\frac{1}{2}} [K cos\theta]] \tag{III.10}$$

where the coordinates (a, θ, η) are respectively the internal radius of the torus, which is constant on the magnetic surface, and the remaining coordinates are the poloidal and toroidal angles. The curvature κ of the magnetic axis depends in general of toroidal coordinate, and the torsion of the magnetic axis is given by τ. In this paper a great simplification would be possible by considering that the torsion and curvature would be constant. These metric allowed Tsypin et al [8] to consider ion viscosity in the plasma. Further ahead we shall also consider the computation of the velocity in terms on the singular metric magnetic surface. The magnetic field which is chosen in the form $B = B_0[1 - \eta_t cos\theta - \eta_h cos(m\theta - n\eta)]$, where η_t is the toroidal amplitude of the magnetic field spectrum and η_h is the helical amplitude, while (m, n) are the poloidal and toroidal modes of the helical magnetic field. Here we shall consider that the B_0 is constant and that the metric component $g_{23} \cong 0$ due to to torsion weakness assumption. Constancy of B_0 also yields g_{13} also vanishes , which turns the plasma helical metric diagonal. Let us now computing the Riemann tensor component

$$R_{1212} = +\frac{\partial^2 g_{12}}{\partial x^1 \partial x^2} - \frac{\partial^2 g_{11}}{\partial x^2 \partial x^2} - \frac{\partial^2 g_{22}}{\partial x^1 \partial x^1} \tag{III.11}$$

which yields for the above metric

$$R_{1212} = \frac{\phi"}{\pi B_0} \tag{III.12}$$

Since we are looking for nested magnetic surfaces which possess constant curvature the helical metric above must satisfy the relation

$$R_{ijkl} = \Lambda(g_{ik}g_{jl} - g_{ij}g_{kl}) \tag{III.13}$$

which in our case yields

$$R_{1212} = \Lambda(g_{11}g_{22} - g_{12}g_{12}) \tag{III.14}$$

or

$$R_{1212} = \frac{\Lambda}{4\pi^2 {B^2}_0} \tag{III.15}$$

Since the curvature component R_{1212} is the same in both expressions (III.12) and (III.16), equating them results in

$$\phi" = \frac{\Lambda}{4\pi B_0} \tag{III.16}$$

By integration one obtains

$$\phi = \frac{\Lambda a^2}{16\pi B_0} + c_1 a + c_2 \tag{III.17}$$

where c_1 and c_2 are integration constants. Now as usual in general relativity we assume that a singularity could be obtained either by the equations $g_{11} = \infty$ and $g_{22} = 0$, as happens as in Schwarzschild static black hole metric solution of Einstein vacuum field equation. This set of equations can be obtained from the helical metric above by setting $\phi(a) = 0$, which from (III.17) yields the following second order algebraic equation in the radius a

$$\frac{\Lambda a^2}{16\pi B_0} + c_1 a + c_2 = 0 \tag{III.18}$$

which solution is

$${a_0}^{\pm} = \frac{8\pi B_0}{\Lambda}[c_1(-1 \pm [1 - \frac{\Lambda}{8\pi B_0}])] \tag{III.19}$$

Thus we are left with two solutions , the simpler of which is ${a_0}^+ = -c_1$ and

$$ {a_0}^- = [1 - \frac{16\pi B_0}{\Lambda}]c_1 \tag{III.20} $$

Note that if $c_1 > 0$ solution a_0 is unphysical because the radius of the helical plasma device cannot be negative, however the physical solution ${a_0}^-$ yields a magnetic field bound from the Ricci scalar

$$ B_0 \leq \frac{\Lambda c_1}{16\pi {c_2}^2} \tag{III.21} $$

To obtain the Ricci collineations we compute first the Ricci scalar components, since the metric is diagonal the only surviving components are

$$ R_{11} = g^{22} R_{l212} = \frac{\Lambda}{4\phi\pi B_0} \tag{III.22} $$

$$ R_{22} = g^{11} R_{l212} = \Lambda\phi \tag{III.23} $$

These equations yields the Ricci scalar as

$$ R = g^{11} R_{11} + g^{22} R_{22} = \Lambda(1 + \pi B_0)\phi \tag{III.24} $$

since in general in plasma devices the field B_0 is very strong we can consider that $\pi {B^2}_0 >> B_0$ which from expressions (III.21) and (III.24) yields an expression which bounds B_0 in terms of the Ricci curvature scalar as

$$ B_0 \leq \frac{\sqrt{c_1 R}}{4\pi c_2} \tag{III.25} $$

This formula can certainly help in the building of new stellarators and toroidal plasma devices in general.

IV Magnetic surface singularities in tokamaks and Ricci collineations

Let us now start by considering the plasma metric given by Zakharov and Shafranov [11] to investigate the evolution of equilibrium of toroidal plasmas. The components g_{ik} (i,k=1,2,3) and (a, θ, z) as coordinates, of their plasma metric are

$$ g_{11} = 1 - 2\Delta' cos\theta + {\Delta'}^2 \tag{IV.26} $$

$$g_{22} = a^2 \tag{IV.27}$$

$$g_{33} = (R - \Delta + acos\theta)^2 \tag{IV.28}$$

$$g_{12} = a\Delta' sin\theta \tag{IV.29}$$

where $z = asin\theta$ and the dash represents derivation with respect to a. In our approximation the last term in the expression (II.1) may be dropped. The magnetic surface equations are tori with circular cross-section and equations

$$r = R - \Delta(a) + acos\theta \tag{IV.30}$$

$$z = asin\theta \tag{IV.31}$$

We shall consider now just two independent coordinates $(x^1 = a, x^2 = \theta)$ since nested surfaces are bidimensional in the case of plasmas, Let us now compute the Riemann tensor components in the linear approximation

$$R_{1212} = +\frac{\partial^2 g_{12}}{\partial x^1 \partial x^2} - \frac{\partial^2 g_{11}}{\partial x^2 \partial x^2} - \frac{\partial^2 g_{22}}{\partial x^1 \partial x^1} \tag{IV.32}$$

Substitution of the plasma metric above into expression (IV.32) yields the expression

$$R_{1212} = [3\Delta' cos\theta - 2] \tag{IV.33}$$

It is easy to show that the components R_{1313} and R_{2323} both vanishes within our approximations. At this point we consider that θ is so small that $sin\theta$ vanishes and $cos\theta = 1$ this simplifies extremely our metric and turns it into a diagonal metric where $g_{12} = 0$ and $g^{bb} = (g_{bb})^{-1}$ $(b = 1, 2)$ and this allows us to compute the components of the Ricci tensor from the Riemann component. But before that let us compute the use the condition that the nested surface is an Einstein space to compute the Riemann component again

$$R_{1212} = \Lambda a^2 [1 - 2\Delta' cos\theta] \tag{IV.34}$$

Since both expressions for the Riemann component R_{1212} must coincide, equating expressions (IV.33) and (IV.34) yields an expression for the derivative of the Shafranov shift Δ as

$$\Delta' = -6[1 - \frac{\Lambda a^2}{6}] \tag{IV.35}$$

Integration of this expression yields the value of the shift in terms of the radius a as

$$\Delta = -6a(1 + \frac{\Lambda a^2}{12}] \quad (IV.36)$$

which satisfies the well-known boundary condition $\Delta(0) = 0$. From these expressions one may also compute $\Delta" = 2\Lambda a$. An important result in plasma physics is that tearing instabilities coming from ion or electron currents possess the shift condition $\Delta' < 0$. This condition would be clearly fulfilled from expression unless the Λ curvature constant would be negative and in modulus $\frac{\Lambda a^2}{2} < -1$. Note that this situation is very similar to the condition of favorable or unfavorable curvature for the instabilities in plasmas [3]. The main difference is that here we are refereeing to Riemann curvature and not to Frenet curvature of the magnetic lines in plasmas. This suggests another method to classify geometrically tearing instabilities. Actually, since has been shown [12] recently that the Riemann tensor in plasmas can be expressed in terms of the Frenet curvature both methods seems to be equivalent. Now let us compute the Ricci components R_{11} and R_{22} from the component R_{1212} by tensor contraction with metric components g^{11} and g^{22}, which results in the expressions

$$R_{11} = \Lambda[1 - 2\Delta'] \quad (IV.37)$$

and

$$R_{22} = -[2 + \Delta'] \quad (IV.38)$$

which in turn yields the expressions

$$\partial_1 R_{11} = -2\Lambda\Delta" \quad (IV.39)$$

and

$$\partial_1 R_{22} = -\Delta" \quad (IV.40)$$

From equations for $i = 1, k = 2$ one obtains

$$\partial_1 \eta_2 = 0 \quad (IV.41)$$

$$\partial_2 \eta_1 = 0 \quad (IV.42)$$

Substitution of these derivatives of the Ricci tensor components into the Ricci collineations equations one obtains the following set of PDE equations

$$2[2 + \Delta']\partial_1 \eta^1 + \eta^1 \Delta" = 0 \quad (IV.43)$$

which yields

$$\eta^a = [1 + \Delta'] \tag{IV.44}$$

Due to constraint $(III.24)$ the only solution for the equation

$$-[2 + \Delta']\partial_2\eta^2 - (1 + \Delta')\Delta" = 0 \tag{IV.45}$$

is $\eta^2 = 0$.This allows us finally to write down the Killing vector as

$$\vec{\eta} = [(1 + \Delta'), 0] \tag{IV.46}$$

the last relation which this vector will have to satisfy $g_{ij}(\eta^j)^2 = 1$ for the modulus of the Killing vector will allow us to determine Δ and in turn from expression (III.17) will allow us to determine the nested surface radius a in terms of the curvature constant Λ. This implies that

$$\Delta' = \frac{3}{10} \tag{IV.47}$$

or $\Delta = \frac{3}{10}a$, which satisfies the well-known boundary condition $\Delta(0) = 0$. Expression (IV.29) yields that $\Delta' > 0$ which shows physically that the tearing instability cannot come from ion or electron currents [2]. Substitution of this result into (III.17) yields

$$a_0 = \sqrt{\frac{19}{15}\Lambda} \tag{IV.48}$$

The pressure now is easily computed from the expression given in reference 3

$$-2\pi c^2 p\frac{a^4}{R} = -6aJ^2\Delta - (1 - \frac{a}{R})J^2 \tag{IV.49}$$

where by using the value of Δ yields

$$p = \frac{RJ^2}{3\pi c^2\Lambda^2} \tag{IV.50}$$

one notes that a singularity in the pressure decreases as the curvature constant increases which agrees with the reasoning that [11] curvature tends to stabilize the plasma.

V Conclusions

In conclusion, we have investigated a method of classification and identification of tearing instability, allowing for example to distinguish between tearing instabilities that comes from ions and electron currents or not, based on the Riemann curvature constant submanifolds as nested surfaces in Einstein spaces. The Killing symmetries are shown also to be very useful in the classification of plasma metrics in the same way they were useful in classifying general relativistic solutions of Einstein's gravitational equations in four-dimensional spacetime [6]. Since as it is well-known [3] the Δ' behaves as $\frac{1}{\delta W}$ near marginal stability, we must conclude that there is a relation between the stability $\delta W > 0$ or instability $\delta W < 0$ and the positive or negative Riemann curvature of the nested surfaces discussed here. Other interesting examples of the utility of the is method is the Ricci collineations investigations of the twisted magnetic flux tubes and the Arnolds metric for the fast dynamo [13, 14, 15]. Though the examples worked here keep some ressemblance to analog gravity models [16] since our black hole analogy does not carry much to the metric since the plasma metric we use is a real plasma metric and not a pseudo-Riemannian plasma metric built from the scalar wave equation.

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